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Category Theoretical Construction of the Figure of States

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We put Mielnik's construction of the convex set of all states of a physical system in the general frame of category theory and give topological details lacking in previous papers on the subject.

INTRODUCTION

The so-called "convex" approach to the foundations of quantum mechanics explores the different structures (topological, geometrical etc.) of the convex set of all states of the physical system. The set of states—the so-called "statistical figure" (Mielnik, 1974)—is the fundamental object in the convex approach. Mielnik (1974, 1980) gives a general recipe for the construction of the "statistical figure." Let us recall the main steps of his construction.

By X we denote the topological manifold of pure states. For example, X may be the solution manifold of the linear or nonlinear fundamental dynamical (wave) equation of the theory.

By F we denote the class of observables which are experimentally defined real continuous functions on X.

The prescriptions for preparing mixed states are represented by the probability measure π on Borel subsets of X, such that all the integrals:

$$\int_X f(x)\, d\pi(x), \qquad f \in F$$

are convergent.

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The set Π of the prescriptions is (up to topological questions) a simplex (Haag and Bannier, 1978).

For any $f \in F$ we can define an observable $\tilde{f}$ on mixed states prepared according to the prescription π by the formula

$$\tilde{f}(\pi) = \int_X f(x)\, d\pi(x)$$

We "see" the mixed states of our system by means of the observables $\tilde{f} \in \tilde{F}$. The limitation in observability introduces an equivalence relation denoted by $R_{\tilde{F}}$ in the set Π. Two states π and π' are called equivalent iff $\tilde{f}(\pi) = \tilde{f}(\pi')$ for each $\tilde{f} \in \tilde{F}$.

Now one constructs the figure of states S as the quotient set $\Pi / R_{\tilde{F}}$ (Mielnik, 1974; Haag and Bannier, 1978). For example, when we assume that the manifold of pure states is the unit sphere in a Hilber space and observables are all real, continuous quadratic forms on X, then the figure of states S is isomorphic with the convex set of density matrices (Mielnik, 1974).

In our paper we put the construction in the general frame of category theory and give the topological details lacking in the Mielnik and Haag-Bannier papers. For a systematic introduction to category theory we send the reader to Semadeni (1971) and Herrlich and Strecker (1973).

DOCTRINES

We will use the following doctrines, which are consequences of the Eilenberg-MacLane program (Semadeni, 1971), in the formulation of Goguen (Goguen et al., 1973):

Doctrine 1. Any species of mathematical structure is represented by a *category*, whose objects "are that structure," and whose *morphisms* "preserve" it. In our case objects represent the sets of states of a physical system and morphisms the dynamical transformation. In general the structure of the set of states (object) should depend on possible dynamics (morphisms) imposed on the states. Category theory language permits one to retain the consistency between structure of the figure of states and dynamics.

Doctrine 2. Any mathematical scheme for constructing objects of one type from objects of another type as data, is represented by a *functor* between the corresponding categories.

Doctrine 3. Any natural construction is represented by an *adjoint functor.* We mean "natural" in the inuitive sense of being "best possible" with respect to some measure and in some context.

We treat category theory as a "language of structure," where one may think of all aspects of a structure as "working together" in a coherent way.

Thus the natural semantics associated to the syntax of category theory seems to be similar to Bohm's philosophy of "wholeness" (Bohm, 1971). The other (meta-) physical aspects of category theory will be analyzed in the subsequent papers of the author.

CONSTRUCTION

Let X be a topological space, whose elements are the pure states of our system. The topology of X should reflect the observable properties of the pure states. In our paper we assume that the space X is compact. It is rather a strong assumption. In some cases one can show that the solution manifolds of nonlinear wave equations form compact sets. In other cases we can make a suitable compactification of the set of pure states (Posiewnik, 1982), adjoining in this way some states which one may call "unphysical" (Gunson, 1967). When we construct the set of states by means of so-called information systems we also obtain the subset of pure states as a compact one (Posiewnik, 1983).

Definition (*Mielnik, 1974*). An *observable* is a continuous function $f: X \to \mathbb{R}^1$, whose values $f(x)$ are interpretable as the statistical averages on various pure states

The dynamical transformations (transmitters) of pure states will be represented by continuous maps $\varphi: X \to X$. Thus the mathematical structure of our sets of states with dynamics corresponds to the structure of the category Comp, in which objects are compact sets and morphisms are continuous transformations of the sets.

Now we would like to make the next heuristic step and to build up a mathematical structure by means of which one can represent the mixed states of the system and the dynamical transformations of the states. According to the Doctrine 3 the natural construction should be given by some adjoint functor. We have a very good candidate for our purpose in case when the set of pure states is a compact one.

Definition. Let X be a compact space.

A *Radon measure* on X is a regular Borel measure (Semadeni, 1971). $\mathcal{M}(X)$ denotes the set of Radon measures on X. $\mathcal{M}(X)$ is a Banach space with the norm defined by

$$\|\mu\| \overset{df}{=} \sup\left\{\left|\int_X f(x)\, d\mu(x)\right|, f \text{ is Borel measurable and } \|f\| \leq 1\right\}$$

Definition. A *Choquet simplex* is a compact convex set K being a base of a cone C such that C generates a lattice order in the space $C - C$.

It can be easily shown that the set

$$\mathcal{S}(X)=\{\mu\in\mathcal{M}(X)\colon \mu\geq 0 \text{ and } \mu(X)=1\}$$

is a Choquet simplex.

Proposition (Semadeni, 1971). Let $\varphi\colon X\to Y$ be a morphism in Comp. If $\mu\in\mathcal{M}(X)$ and B is a Borel subset of Y, define $\nu(B)=\mu(\overleftarrow{\varphi}(B))$. Then ν is a Radon measure on Y. Moreover the map $\Phi\colon\mathcal{M}(X)\to\mathcal{M}(Y)$ defined as $\Phi(\mu)=\nu$ for μ in $\mathcal{M}(X)$ is a nonnegative linear operator, and $\|\Phi\|\geq 1$.

We say that Φ is induced by φ and denote it by $\mathcal{M}(\varphi)$. Let Ban_1 be the category of Banach spaces and linear contractions.

Then

$$\mathcal{M}\colon \text{Comp}\to\text{Ban}_1$$

is a covariant functor called the *Radon functor.*

Let $\mathcal{S}(\varphi)$ denote the restriction of $\mathcal{M}(\varphi)\colon\mathcal{M}(X)\to\mathcal{M}(Y)$ to $\mathcal{S}(X)$. It can be easily shown that $\mathcal{S}(\varphi)\colon\mathcal{S}(X)\to\mathcal{S}(Y)$ is a continuous affine map and $\mathcal{S}\colon\text{Comp}\to\text{Compconv}$ (Compconv is the category of compact convex sets and continuous affine maps) is a covariant functor.

We may call it the *simplex functor.*

Moreover $\mathcal{S}$ is a left-adjoint to the forgetful functor: $\square$: Compconv→ Comp and the map $\delta\colon X\to\mathcal{S}(X)$ where $\delta(x)$, $x\in X$ is Dirac measure concentrated on the set $\{x\}$, determines the corresponding natural transformation.

Now we can in accord with the Doctrine 3 postulate that the construction of the mathematical structure of the set of prescriptions for preparing the mixed states is provided by the simplex functor $\mathcal{S}$. The transmitters corresponding to the morphisms $\varphi\colon X\to X$ are now given by continuous affine transformations $\mathcal{S}(\varphi)$ of the set of probability Radon measures on X.

The other properties of the construction resulting from the category theoretical qualities of the functor $\mathcal{S}$ will be investigated in the subsequent papers of the author. For any observable f we can define the corresponding observable $\tilde{f}$ on the set $\mathcal{S}(X)$ by the formula

$$\tilde{f}(\mu)=\int_X f(x)\,d\mu(x),\qquad \mu\in\mathcal{S}(X)$$

The integral is convergent for all continuous functions on X. The observables $\tilde{f}$ form a set $\tilde{F}$.

Let us denote by $l(X)$ the space of all continuous functions on the set X and by $l^*(X)$ the space of linear continuous functionals on the space $l(X)$.

If X is compact then the formula $\int_X g(x)\,d\mu(x)$ defines for any Radon measure $\mu\in\mathcal{M}(X)$ a linear functional on $l(X)$, which we denote as $\theta(\mu)$.

Then one can show (Semadeni, 1971) that

$$\theta: \mathscr{S}(X) \to l^*(X)$$

is an affine isometrical bijection.

In this way every observable $\tilde{f} \in \tilde{F}$ is an affine continuous function on the set $\mathscr{S}(X)$.

SUMMARY

The set of prescriptions for preparing the mixed states of our system is the set $\mathscr{S}(X)$ of probability Radon measures on the (compact) set of pure states X.

Observables are affine continuous functions on the set $\mathscr{S}(X)$ and dynamical transformations (transmitters) are given by continuous affine contractions $\mathscr{S}(\varphi): \mathscr{S}(X) \to \mathscr{S}(X)$. Now let us define, in the same way as in the Introduction, the equivalence relation $R_{\tilde{F}}$ on the set $\mathscr{S}(X)$.

$R_{\tilde{F}}$-equivalent probability measures represent physically indistinguishable mixed states if the only way of "seeing" them is by means of observables from the class $\tilde{F}$.

Using the recipe of Mielnik we obtain the statistical figure S as a quotient set: $\mathscr{S}(X)/R_{\tilde{F}}$. The set $\mathscr{S}(X)$ has an innate natural affine structure. Because all observables from the class $\tilde{F}$ are affine, the structure is consistent with the relation $R_{\tilde{F}}$ and we can transfer the structure to the quotient set $S = \mathscr{S}(X)/R_{\tilde{F}}$. The statistical figure S becomes in this way a convex set. Let q be the natural quotient map, $q: \mathscr{S}(X) \to S = \mathscr{S}(X)/R_{\tilde{F}}$, $q(\mu) = [\mu]$, the $R_{\tilde{F}}$-equivalence class of the measure $\mu \in \mathscr{S}(X)$.

We can define observables on the set S by means of the prescription

$$\bar{f}[\mu] \overset{df}{=} \tilde{f}(\mu), \qquad \tilde{f} \in \tilde{F}$$

where μ is any element from the equivalence class $[\mu]$. It is a standard result that the definition is consistent, i.e., value of the observable $\bar{f}$ does not depend on the choice of an element μ from the equivalence class $[\mu]$ and we can write $\bar{f} = \tilde{f}q^{-1}$.

Also it is obvious that each observable $\bar{f}$ is an affine function on the set S. We equip the set $S = \mathscr{S}(X)/R_{\tilde{F}}$ with the quotient topology. The map q is continuous in the topology. Moreover, if we assume that the quotient topology is a Hausdorff one, then S is a compact set as a continuous image of the compact set $\mathscr{S}(X)$.

Observables $\bar{f}$ are continuous maps because they are bounded on S.

Definition. Let A be a set and R an equivalence relation on A. A map $T: A \to A$ is said to be *compatible with R* iff aRa' implies $TaRTa'$ for any

$a, a' \in A$. Then there exists a map $\bar{T}: A/R \to A/R$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{T} & A \\ \downarrow q & & \downarrow q \\ A/R & \xrightarrow{\bar{T}} & A/R \end{array}$$

is commutative.

$\bar{T}$ is defined as $\bar{T}([a]_R) \stackrel{df}{=} [T(a)]_R$. We can ask whether the transmitters $\mathscr{S}(\varphi)$ are compatible with the relation $R_{\tilde{F}}$. In that case we could define the dynamical transformations $\overline{\mathscr{S}(\varphi)}$ of the figure of states S. Let us return for a while to the space X of the pure states. Now Mielnik (1974) says the following:

> Indeed having any measuring device destined to measure a certain statistical average $f \in F$, one can produce more observables by altering the measurement process. Instead of measuring straightforwardly the statistical average f on a given wave $x \in X$ one can let x undergo first a certain preliminary kineto-dynamical process [in our denotations] φ and only afterwards measure f on the evolved wave $x' = \varphi(x)$, thus obtaining a new statistical quantity $(f\varphi)(x) = f(x') = f(\varphi(x))$. In this way the existence of "motions" prevents one from assuming too poor a class of observables: having any observable $f \in F$, one must also assume the existence in F of the infinity of other observables of the form $f\varphi$ generated by all possible evolution processes which the system might perform under the influence of various external forces.

This suggests the admission in our formalism of the following assumption:

Assumption. The class F is invariant with respect to all dynamical morphisms $\varphi: X \to X$.

Now let $\mu \in \mathscr{S}(X)$. If $\nu = \mathscr{S}(\varphi)\mu$ then for every g in $l(x)$ the following substitution rule holds (Semadeni, 1971):

$$\int_X g(x)\, d\nu(x) = \int_X g(\varphi(x))\, d\mu(x)$$

Let $f \in F$ be an observable and $\varphi: X \to X$ a morphism in Comp. Then from the Assumption it follows that $f \cdot \varphi$ is also an observable.

Observables f and $f\varphi$ generate observables $\tilde{f}$ and $\widetilde{f\varphi}$; respectively, on the set $\mathscr{S}(X)$.

From the substitution rule it follows that

$$(\widetilde{f\varphi})(\mu) = \tilde{f}(\mathscr{S}(\varphi)\mu)$$

for each $\mu \in \mathcal{S}(X)$. In this way we see that the class $\tilde{F}$ of observables on $\mathcal{S}(X)$ is invariant with respect to all morphisms $\mathcal{S}(\varphi)$. In that case one can easily show that each transmitter is compatible with the relation $R_{\tilde{F}}$ and thus it gives rise to an (affine) transmitter $\overline{\mathcal{S}(\varphi)}$ on the statistical figure S. $\overline{\mathcal{S}(\varphi)} = q\mathcal{S}(\varphi)q^{-1}$.

Remark. What happens when there is a transmitter $\mathcal{S}(\varphi)$ which is not compatible with $R_{\tilde{F}}$ (so our Assumption is not true)? Then in general we have splitting of the equivalence classes of $R_{\tilde{F}}$. Two states which were indistinguishable by observables from the class $\tilde{F}$ after the evolution by means of $\mathcal{S}(\varphi)$ become distinguishable.

So we may say that the same initial states, under the same conditions, give rise to two distinct states. In this situation we are dealing with acausality, which we must interpret on stochastic grounds. When $\mathcal{S}(\varphi)$ is compatible with $R_{\tilde{F}}$ but $\mathcal{S}(\varphi)^{-1}$ is not, then we deal with irreversibility—because $\mathcal{S}(\varphi)$ may map two or more distinct classes into a common $R_{\tilde{F}}$ class. End of remark.

Lemma (Engelking, 1975). A mapping T of the quotient space W/R into a topological space Z is continuous iff the composition Tq is continuous (q is the natural quotient map).

From the Lemma we get that each transmitter $\overline{\mathcal{S}(\varphi)}$ on S is continuous because $\overline{\mathcal{S}(\varphi)}q = q\mathcal{S}(\varphi)$, but q and $\mathcal{S}(\varphi)$ are continuous.

Now let us change the topological structure in the set S. The new topology will be the so-called $\sigma(S, \bar{F})$ topology, a base for which there is a family of sets of the form

$$\mathcal{U}(s_0, \bar{f}_1, \ldots, \bar{f}_n; \varepsilon_1, \ldots, \varepsilon_n) = \bigcap_{i=1}^{n} \{s \in S: |\bar{f}_i(s) - \bar{f}_i(s_0)| < \varepsilon_i\} \qquad (1)$$

where $s_0 \in S$, $\varepsilon_i > 0$, $i = 1, 2, \ldots, n$, and $\{\bar{f}_1, \ldots, \bar{f}_n\}$ is any finite subset of $\bar{F}$.

The $\sigma(S, \bar{F})$ topology is a very natural one and is particularly appropriate for expressing the usual limitations of any real experiment. $\sigma(S, \bar{F})$ neighborhood (1) can be interpreted as consisting of all states which cannot be distinguished from s_0 when values of observables $\bar{f}_i$ are determined with errors ε_i.

Because all $\bar{f}$ from $\bar{F}$ are affine and the class $\bar{F}$ distinguishes points of S the $\sigma(S, \bar{F})$ topology is locally convex and Hausdorff.

We have the following:

Proposition (Semadeni, 1971). Let (W, τ) be a compact space and let G be any subset of $l(W)$ separating W. Then $\tau = \sigma(W, G)$, i.e., the original topology of W and the $\sigma(W, G)$ topology are identical. From the Proposition we get that the figure of states S is a compact convex set in locally convex Hausdorff topology $\sigma(S, \bar{F})$.

SUMMARY

The figure of states S of a physical system with compact manifold of pure states is a compact convex set in locally convex Hausdorff topology.

The class of observables consists of continuous affine functions on S.

The dynamical transformations are continuous affine maps of the set S into itself.

It is a good starting point for the applications of the Choquet theory (Alfsen, 1971) which we investigate in the subsequent papers (Posiewnik and Pykacz, 1983).

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REFERENCES

Alfsen, E. M. (1971). *Compact Convex Sets and Boundary Integrals*, Ergwbnisse der math. BD.57 (Springer-Verlag, Berlin).

Bohm, D (1971). *Found. Phys.*, **1**, 359.

Engelking, R. (1975). *Topologia ogólna* (PWN, Warsaw) (in Polish).

Goguen, J. A., Thatcher, J. W., Wagner, E. G., and Wright, J. B. (1973). A junction between computer science and category theory: I, Part 1, IBM Research Report RC 4526.

Gunson, J. (1967). *Commun. Math. Phys.* **6**, 262.

Haag, R., and Bannier, U (1978). *Commun. Math. Phys.*, **60**, 1.

Herrlich, H., and Strecker, G. E. (1973). *Category Theory* (Allyn and Bacon, Boston).

Mielnik, B. (1974). *Commun. Math. Phys.* **37**, 221.

Mielnik, B. (1980). *J. Math. Phys.* **21**, 44.

Posiewnik, A. (1982). On a compactification of the set of states of physical system (preprint).

Posiewnik, A. (1983). Applications of information systems in convex approach to physical systems (preprint).

Posiewnik, A., and Pykacz, J. (1983). Choquet properties of the sets of physical states (preprint).

Semadeni, Z. (1971). Banach spaces of continuous functions (PWN, Warsaw).